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Stability of Switched and Hybrid Systems*

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Abstract

This paper outlines some preliminary work on the stability analysis of switched and hybrid systems. The hybrid systems considered are those that combine continuous dynamics—represented by differential or difference equations—with finite dynamics—usually thought of as being a finite automaton. Here, we concentrate on the continuous dynamics and model the finite dynamics as switching among finitely many continuous systems. We introduce multiple Lyapunov functions as a tool for analyzing Lyapunov stability of such "switched systems." We use IFS theory as a tool for Lagrange stability. We also disuss the case where the switched systems are indexed by an arbitrary compact set.

1 Introduction

We have in mind the following model as a prototypical example of a *switched system*:

$$\dot{x}(t) = f_i(x(t)), \qquad i \in \{1, \ldots, N\}, \tag{1}$$

where $x(t) \in \mathbb{R}^n$. We add the following assumptions.

- Each f_i is globally Lipschitz continuous.
- The i's are picked in such a way that there are finite switches in finite time.

Such systems are of "variable structure" or "multimodal"; they are a simple model of (the continuous portion) of hybrid systems. We explain this below. The particular i at any given time may be chosen by some "higher process," such as a controller, computer, or human operator, in which case we say that the system is *controlled*. It may also be a function of time or state or both, in which case we say that the system is *autonomous*. In the latter case, we may really just arrive at a single (albeit complicated) nonlinear, time-varying equation. However, one might gain some leverage in the analysis of such systems by considering them to be amalgams of simpler systems.

We also discuss difference equations

$$x[k+1] = f_i(x[k+1]), \qquad i \in \{1, \ldots, N\}, \tag{2}$$

where $x[k] \in \mathbb{R}^n$. Here, we only add the assumption that each f_i is globally Lipschitz continuous. Again, these equations can be thought of as the "continuous" portion of the dynamics of hybrid systems combining difference equations and finite automata [7].

Models like Eq. (1) have been studied for stability [9, 17]. We build on some of their notation. However, those papers were predominantly concerned with the case where all the f_i are linear.

The paper is organized as follows. The next section discusses hybrid systems as motivation for the study of stability of switched systems. It may be skipped with no loss of continuity. Section 3 introduces "multiple Lyapunov functions" as a tool for analyzing Lyapunov stability of switched systems. In section 4, iterative function systems are presented as a tool for proving Lagrange stability and positive invariance. We also address the case where $\{1, \ldots, N\}$ in Eqs. (1) and (2) is replaced by an arbitrary compact set. We conclude with some discussion.

Throughout, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$, $\mathbb{Z}^+$ denote the reals, nonnegative reals, integers, and nonnegative integers, respectively.

2 Motivation from Hybrid Systems

Hybrid systems are those that inherently combine logical and continuous processes, usually coupled finite automata and differential equations [2, 6, 7, 11, 12, 16, 18]. Thus, the continuous dynamics is modeled by a differential equation

$$\dot{x}(t) = \xi(t), \qquad t \geq 0, \tag{3}$$

where $x(t)$ is the *continuous component* of the state taking values in some subset of a Euclidean space. $\xi(t)$ is a (controlled) vector field that generally depends on $x(t)$ and the aforementioned "logical" or "finite" dynamics.

As mentioned above, we consider two categories of switched systems [6]:

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- *Autonomous switching.* Here the vector field $\xi(\cdot)$ changes discontinuously when the state $x(\cdot)$ hits certain "boundaries."
- *Controlled switching.* Here $\xi(\cdot)$ changes abruptly in response to a control command, possibly with an associated cost.

A (continuous-time) *autonomous hybrid system* may be defined as follows:

$$\begin{aligned} \dot{x}(t) &= f(x(t), q(t)), \\ q(t) &= \nu(x(t), q(t^-)), \end{aligned} \tag{4}$$

where $x(t) \in \mathbb{R}^n$, $q(t) \in Q \simeq \{1, \ldots, N\}$. Here, $f(\cdot, q) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $q \in Q$, each globally Lipschitz continuous, is the *continuous dynamics* of Eq. (4); and $\nu : \mathbb{R}^n \times Q \longrightarrow Q$ is the *finite dynamics* of Eq. (4). Here, the notation t^- indicates that the finite state is piecewise continuous from the right. Thus, starting at $[x_0, i]$, the continuous state trajectory $x(\cdot)$ evolves according to $\dot{x} = f(x, i)$. If $x(\cdot)$ hits some $(\nu(\cdot, i))^{-1}(j)$ at time t_1, then the state becomes $[x(t_1), j]$, from which the process continues.

Clearly, this is an instantiation of autonomous switching. Switchings that are a fixed function of time may be taken care of by adding another state dimension, as usual. This definition is closely related to the so-called differential automata in [18]; it is a simplified view of the hybrid systems models in [2, 6, 7, 16]. We do not discuss here restrictions on ν which lead to finite switches in finite time. For a discussion of this, see [3, 6, 7, 18].

By a (continuous-time) *controlled hybrid system* we have in mind a system of the form:

$$\begin{aligned} \dot{x}(t) &= f(x(t), q(t), u(t)), \\ q(t) &= \nu(x(t), q(t^-), u(t)), \end{aligned} \tag{5}$$

where everything is as above except that $u(t) \in \mathbb{R}^m$, with f and ν modified appropriately.

Next we give an example, in which we have suppressed the finite dynamics:

Example 1 *A simplified model of a manual transmission is given by* [7]

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= [-a(x_2) + u]/(1 + v), \end{aligned}$$

where x_1 *is the ground speed,* x_2 *is the engine RPM,* $u \in [0, 1]$ *is the throttle position, and* $v \in \{1, 2, 3, 4\}$ *is the gear shift position. The function* a *is positive for positive argument.*

Likewise, we can define discrete-time autonomous and controlled hybrid systems by replacing the ODEs above with difference equations. In this, case Eq. (4) represents a simplified view of some of the models in [7].

3 Multiple Lyapunov Functions

In this section, we discuss Lyapunov stability of switched systems via "multiple Lyapunov functions." The idea here is that even if we have Lyapunov functions for each system f_i individually, we need to impose restrictions on switching to guarantee stability. Indeed, it is easy to construct examples of two globally exponentially stable systems and a switching scheme that sends all trajectories to infinity:

Example 2 *Consider* $f_1(x) = Ax$ *and* $f_2(x) = Bx$ *where*

$$A = \begin{bmatrix} -0.1 & 1 \\ -10 & -0.1 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.1 & 10 \\ -1 & -0.1 \end{bmatrix}.$$

Then $\dot{x} = f_i(x)$, *is globally exponentially stable for* $i = 1, 2$. *But the switched system using* f_1 *in the second and fourth quadrants and* f_2 *in the first and third quadrants is unstable. See Figures 1-3, which plot ten seconds of trajectories for* f_1, f_2, *and the switched system starting from* $(1,0)$, $(0,1)$, $(10^{-6}, 10^{-6})$, *respectively.*

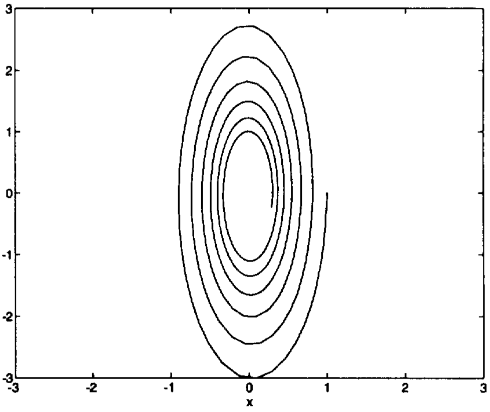

Figure 1: f_1 system trajectory.

We assume the reader is familiar with basic Lyapunov theory (continuous and discrete time), say, at the level of [14]. The level of rigor of the proofs is similar to those in that book. We let $S(r)$, $B(r)$, and $\overline{B}(r)$ represent the sphere, ball, and closed ball of Euclidean radius r about the origin in $\mathbb{R}^n$, respectively.

Below, we will be dealing with systems that switch among vector fields (resp. difference equations), over time or regions of state-space. One can associate with such a system the following *(anchored) switching sequence*, indexed by an initial state, x_0:

$$S = x_0; \ (i_0, t_0), (i_1, t_1), \ldots, (i_N, t_N), \ldots. \tag{6}$$

The sequence may or may not be infinite. In the finite case, we may take $t_{N+1} = \infty$, with all further

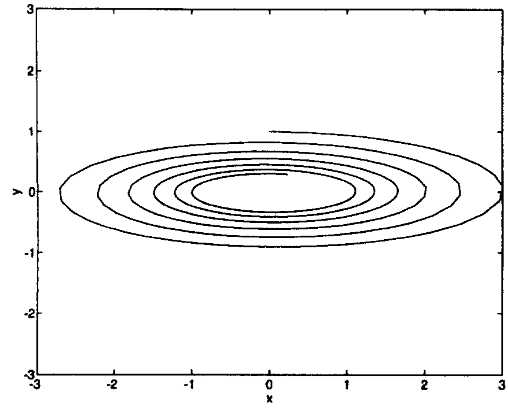

Figure 2: f_2 system trajectory.

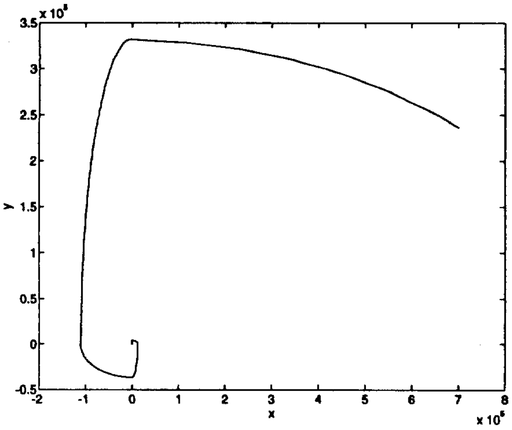

Figure 3: Switched system trajectory.

definitions and results holding. However, we present in the sequel only in the infinite case to ease notation. The switching sequence, along with Eq. (1), completely describes the trajectory of the system according to the following rule: (i_k, t_k) means that the system evolves according to $\dot{x}(t) = f_{i_k}(x(t), t)$ for $t_k \leq t < t_{k+1}$. We denote this trajectory by $x_S(\cdot)$. Throughout, we assume that the switching sequence is *minimal* in the sense that $i_j \neq i_{j+1}$, $j \in \mathbb{Z}^+$.

We can take projections of this sequence onto its first and second coordinates, yielding the sequence of indices,

$$\pi_1(S) = x_0;\ i_0, i_1, \ldots, i_N, \ldots,$$

and the sequence of switching times,

$$\pi_2(S) = x_0;\ t_0, t_1, \ldots, t_N, \ldots,$$

respectively. Suppose S is a switching sequence as in Eq. (6). We will denote by $S|i$ the sequence of switching times whose corresponding index is i. The *interval completion* $\mathcal{I}(T)$ of a strictly increasing sequence of times $T = t_0, t_1, \ldots, t_N, \ldots$, is the set

$$\bigcup_{j \in \mathbb{Z}^+} (t_{2j}, t_{2j+1}).$$

Finally, let $\mathcal{E}(T)$ denote the *even sequence* of T:

$$t_0, t_2, t_4, \ldots.$$

Below, we say that V is a *candidate Lyapunov function* if V is a continuous, positive definite function (about the origin, 0) with continuous partial derivatives. Note this assumes $V(0) = 0$. We also use

Definition 3 (Lyapunov-like) *Given a strictly increasing sequence of times T in $\mathbb{R}$* (resp. $\mathbb{Z}$)*, we say that V is a* Lyapunov-like function for function f and trajectory $x(\cdot)$ (resp. $x[\cdot]$) over T *if*

- $\dot{V}(x(t)) \leq 0$ (resp. $V(x[t+1]) \leq V(x[t])$) *for all $t \in \mathcal{I}(t)$* (resp. *$t \in T$*),
- *V is monotonically nonincreasing on $\mathcal{E}(T)$* (resp. *T*).

Theorem 4 *Suppose we have candidate Lyapunov functions V_i, $i = 1, \ldots, N$, and vector fields $\dot{x} = f_i(x)$* (resp. *difference equations $x[k+1] = f_i(x[k])$*) *with $f_i(0) = 0$, for all i. Let S be the set of all switching sequences associated with the system.*

If for each $S \in \mathcal{S}$ we have that for all i, V_i is Lyapunov-like for f_i and $x_S(\cdot)$ over $S|i$, then the system is stable in the sense of Lyapunov.

Proof In each case, we do the proofs only for $N = 2$.

- *Continuous-time:* Let $R > 0$ be arbitrary. Let $m_i(\alpha)$ denote the minimum value of V_i on $S(\alpha)$. Pick $r_i < R$ such that in $B(r_i)$ we have $V_i < m_i(R)$. This choice is possible via the continuity of V_i. Let $r = \min(r_i)$. With this choice, if we start in $B(r)$, either vector field alone will stay within $B(R)$.

  Now, pick $\rho_i < r$ such that in $B(\rho_i)$ we have $V_i < m_i(r)$. Set $\rho = \min(\rho_i)$. Thus, if we start in $B(\rho)$, either vector field alone will stay in $B(r)$. Therefore, whenever the other is first switched on we will have $V_i(x(t_1)) < m_i(R)$, so that we will stay within $B(R)$.

- *Discrete-time:* Let $R > 0$ be arbitrary. Let $m_i(\alpha, \beta)$ denote the minimum value of V_i on the closed annulus $\overline{B}(\beta) - B(\alpha)$. Pick $R_0 < R$ so that none of the f_i can jump out of $B(R)$ in one step. Pick $r_i < R_0$ such that in $B(r_i)$ we have $V_i < m_i(R_0, R)$. This choice is possible via the continuity of V_i. Let $r = \min(r_i)$. With this choice, if we start in $B(r)$, either equation alone will stay within $B(R)$.

Pick $r_0 < r$ so that none of the f_i can jump out of $B(r)$ in one step. Now, pick $\rho_i < r_0$ such that in $B(\rho_i)$ we have $V_i < m_i(r_0, r)$. Set $\rho = \min(\rho_i)$. Thus, if we start in $B(\rho)$, either equation alone will stay in $B(r_0)$, and hence $B(r)$. Therefore, whenever the other is first switched on we will have $V_i(x(t_1)) < m_i(R_0, R)$, so that we will stay within $B(R_0)$, and hence $B(R)$.

The proofs for general N require N sets of concentric circles constructed as the two were in each case above. □

Some remarks are in order:

- The case $N = 1$ is the usual theorem for Lyapunov stability [14]. Also, compare Figures 4 and 5, both of which depict the continuous-time case.
- The theorem also holds if the f_i are time-varying.
- It is easy to see that the theorem does not hold if $N = \infty$, and we leave it to the reader to construct examples.

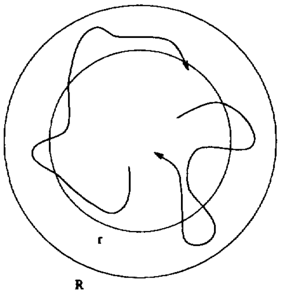

Figure 4: Cartoon of Lyapunov stability.

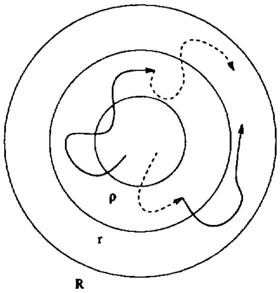

Figure 5: Cartoon of multiple Lyapunov stability, $N = 2$.

It is possible to use different conditions on the V_i to ensure stability. For instance, consider the following

Definition 5 *If there are candidate Lyapunov functions V_i corresponding to f_i for all i, we say they satisfy the* sequence nonincreasing condition for a trajectory $x(\cdot)$ *if*

$$V_{i_{j+1}}(x(t_{j+1})) < V_{i_j}(x(t_j)).$$

This is a stronger notion than the Lyapunov-like condition used above.

The sequence nonincreasing condition is used in the stability (version of the asymptotic stability) theorem of [17]. Thus that theorem is a special case of the continuous-time version of Theorem 4 above. Moreover, the proof of asymptotic stability in [17] is flawed since it only proves state convergence and not state convergence plus stability, as required. It can be fixed using our theorem.

Now, consider the case where the index set is an arbitrary compact set:

$$\dot{x} = f(x, \lambda), \qquad \lambda \in K, \text{ compact.} \tag{7}$$

Here, $x \in \mathbb{R}^n$ and f is globally Lipschitz in x, continuous in λ. For brevity, we only consider the continuous-time case. Again, we assume finite switches in finite time.

As above, we may define a switching sequence

$$S = x_0;\ (\lambda_0, t_0), (\lambda_1, t_1), \ldots, (\lambda_N, t_N), \ldots.$$

with its associated projection sequences.

Theorem 6 *Suppose we have candidate Lyapunov functions $V_\lambda \equiv V(\cdot, \lambda)$ and vector fields as in Eq. (7) with $f(0, \lambda) = 0$, for each $\lambda \in K$. Also, $V : \mathbb{R}^n \times K \longrightarrow \mathbb{R}^+$ is continuous. Let S be the set of all switching sequences associated with the system.*

If for each $S \in \mathcal{S}$ we have that for all i, V_λ is Lyapunov-like function for f_λ and $x_S(\cdot)$ over $S|\lambda$, and the V_λ satisfy the sequence nonincreasing condition for $x_S(\cdot)$, then the system is stable in the sense of Lyapunov.

Proof We present the proof in the case that K is sequentially compact, which is automatic if K is a metric space. The general case follows with little change from the argument below by using countable compactness and nets instead of sequences. (See [10, 15] for definitions).

The Lyapunov-like and sequence nonincreasing constraints are such that if $\pi_i(S) = x_0;\ \lambda_0, \lambda_1, \lambda_2, \ldots$, then the state $x(t)$ will remain within the set:

$$R_{V(x_0,\lambda_0)} \equiv \bigcup_{\lambda \in K} \{x \mid V(x, \lambda) < V(x_0, \lambda_0)\}.$$

Next, note that if x_0 lies in

$$I_\epsilon \equiv \left\{ x \mid \sup_{\lambda \in K} V(x, \lambda) < \epsilon \right\},$$

then the state will remain in R_ϵ.

Thus, it remains to show that given any $\epsilon > 0$, there exist $\epsilon', \delta > 0$ such that

$$B(\delta) \subset I_{\epsilon'} \cap B(\epsilon) \subset R_{\epsilon'} \cap B(\epsilon) \subset B(\epsilon).$$

Letting m denote the minimum of V on $S(\epsilon) \times K$, $\epsilon' = m/2$ satisfies the last equation. Now $I_{\epsilon'}$ contains the origin, $0 \in \mathbb{R}^n$. Suppose there is no open ball about 0 in $I_{\epsilon'}$. Then for each $n \in \mathbb{Z}^+$, there exists y_n such that

$$\|y_n\| \leq 1/n, \qquad \sup_{\lambda \in K} V(y_n, \lambda) \geq \epsilon'.$$

Further, we may take each of the y_n distinct. Let $\lambda_n \in K$ be the point at which the sup above is attained. Since $\overline{B}(\epsilon') \times K$ is sequentially compact, there is a subsequence $\{(y_{i_k}, \lambda_{i_k})\}$ converging to $(0, \lambda^*)$ with $V(y_{i_k}, \lambda_{i_k}) \geq \epsilon'$, a contradiction to the continuity of V and the assumption that $V(0, \lambda) = 0$ for all $\lambda \in K$. □

This theorem is a different generalization of the aforementioned theorem in [17].

4 Iterated Function Systems

In this section, we study IFS theory as a tool for Lagrange stability. We begin with some background from [4, 20, 8]:

Definition 7 (IFS) *Recall that a* contractive function f *is one such that there exists* $s < 1$ *where* $d(f(x), f(y)) \leq s d(x, y)$, *for all* x, y.

An IFS (iterated function system) *is a complete metric space and a set* $\{f_i\}_{i \in I}$ *of contractive functions such that* I *is a compact space and the map* $(x, i) \mapsto f_i(x)$ *is continuous.*

The image of a compact set X under an IFS is the set $Y = \bigcup_{i \in I} f_i(X)$. It is compact. Now suppose W is an IFS. Let $S(W)$ be the semi-group generated by W under composition. For example, if $W = \{f, g\}$ then

$$S(W) = f, g, f \circ f, f \circ g, g \circ f, g \circ g, \ldots.$$

Now, define A_W to be the closure of the fixed points of $S(W)$. We have

Theorem 8 *Suppose* $W = \{w_i\}_{i \in I}$ *is an IFS on* X. *Then*

- A_W *is compact.*
- $A_W = \bigcup_{i \in I} w_i(A_W)$.
- *For all* $x \in X$,

  $$A_W = \bigcup_{\sigma} \left\{ \lim_{n \to \infty} w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(x) \right\},$$

  where $\sigma = (\sigma_1, \sigma_2, \ldots)$, $\sigma_i \in I$.

The relevance of this theorem is twofold:

- A_W is an invariant set under the maps $\{w_i\}_{i \in I}$.
- All points approach A_W under iterated composition of the maps $\{w_i\}_{i \in I}$.

Clearly, this theory can be applied in the case of a set of contractive discrete maps indexed by a compact set (usually finite). Thus, it is directly applicable to systems of the form Eq. (2)

To obtain contractive maps while switching among differential equations requires a little thought. Assume there is some lower limit T on the inter-switching time. Now, notice that for any inter-switching time $r \geq T$, there is a decomposition into smaller intervals as follows:

$$r = \sum_{i=1}^{M} t_i, \qquad t_i \in [T, 2T].$$

Proof Let $k = \lfloor r/(2T) \rfloor$ and $q = r - 2Tk$. Now, $2T > q \geq 0$. If $q = 0$, the decomposition is $t_i = 2T$, $i = 1, \ldots, k$. If $2T > q \geq T$, the decomposition is $t_i = 2T$, $i = 1, \ldots, k$; $t_{k+1} = q$; the first equation not applying if $k = 0$. Finally, if $T > q > 0$, then (we must have $k \geq 1$ since $r \geq T$) and $2T > q + T > T$, so the decomposition is $t_i = 2T$, $i = 1, \ldots, k-1$; $t_k = T$; $t_{k+1} = T + q$; the first equation not applying if $k = 1$. □

Therefore, we can convert switching among vector fields into an IFS by letting $I = \bigcup_{j=1,\ldots,N} j \times [T, 2T]$. In particular, we see that for each i, if it is active for a time $r \geq T$, we can write the solution in that interval as $\phi_r^i(x) = (\circ_{j=1}^{M} \phi_{t_j}^i)(x)$, where ϕ_t^i is the fundamental solution for f_i acting for time t. Thus the switching sequence can be converted to an iterated composition of maps indexed by the compact set I.

The other interesting point about IFS theory is that the different vector fields (or difference equations) need not have the same equilibrium point. This is important as it appears to be the usual case in switched and hybrid systems (cf. Example 1).

In conclusion, in IFS we have a tool for analyzing the Lagrange stability and computing the invariant sets of switched systems of the form Eqs. (1) and (2). The resulting sets A_W are reminiscent of those for usual IFS (see [4]), although we don't give any here. The reader may consult [4] for algorithms to compute such invariant sets.

5 Conclusion

First we discussed hybrid systems. Abstracting away the finite dynamics, we arrived at the concept of a "switched system." Section 3 introduces "multiple Lyapunov functions" (MLFs) as a tool for analyzing Lyapunov stability of switched systems. In section 4, iterative function systems (IFS) were presented as a tool for proving Lagrange stability and positive invariance of such systems.

We now turn to some discussion. Our abstracting away of the finite dynamics to prove stability properties is related to "verification by successive approximation" [1]. In both the MLF and IFS cases, the stability results are sufficiency conditions on the continuous dynamics and switching. This work, which was begun in [5], represents the rudiments of a stability theory of the systems in Eqs. (1) and (2) and, in turn, of hybrid systems. We also discussed the case where $\{1, \ldots, N\}$ in Eqs. (1) and (2) is replaced by an arbitrary compact set.

For future directions, we offer the following brief treatment. In searching for necessary and sufficient stability criteria, we expect that the theory in [13] will be helpful. An early use of our MLF theory is given in [21], which deals with convergence of a combined scheme for robotic planning and obstacle avoidance. As far as IFS, we have yet to explore their full potential. For instance, we can state IFS theorems analogous to Theorem 4, namely, in which the maps need only be contractions on the points (time periods) on which they are applied. Finally, if there is no lower limit T on the inter-switching time, then we are not assured to have a contraction mapping. However, as long as we have only finite switches in finite time, one expects that the trajectories should be well-behaved.

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